On affine root systems in my papers published 2024 or before

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This note aims to clarify some potential misunderstandings regarding choices related to affine root systems in some of my publications. It also explains how to link the results to some other literature.

The technical introductions of the following articles all follow the same pattern:

- Understanding affine Deligne–Lusztig varieties via the quantum Bruhat graph (PhD thesis, TUM)
- Generic Newton points and cordial elements (preprint 2022)
- Affine Bruhat order and Demazure products (preprint 2022, published 2024 in Forum of Mathematics, Sigma)
- Machine learning assisted exploration for affine Deligne-Lusztig varieties, joint with Bin Dong, Xuhua He, Pengfei Jin and Qingchao Yu (preprint 2023, published 2024 in Peking Mathematical Journal)
- Affine Deligne-Lusztig varieties via the double Bruhat graph I: Semi-infinite orbits (preprint 2023)
- Affine Deligne-Lusztig varieties via the double Bruhat graph II: Iwahori-Hecke algebra (preprint 2023)
- Affine Deligne-Lusztig varieties of positive Coxeter type, joint with Ryosuke Shimada and Qingchao Yu (preprint 2023)

In each case, a reductive group G is studied, which is defined over a non–archimedian local field F. We denote the completion of the maximal unramified extension by $\breve{F}$, and the Frobenius by σ.

The choice of a maximal $\breve{F}$-split torus $S \subseteq G_{\breve{F}}$ and its centralizer $T \subseteq G_{\breve{F}}$ yields an apartment of the Bruhat-Tits building $\mathcal{A}$ of $G_{\breve{F}}$. We pick a σ-stable alcove $\mathfrak{a}$ inside $\mathcal{A}$ and denote by $I \subseteq G(\breve{F})$ the stabilizer of this alcove, called our Iwahori subgroup. Each

affine root $a \in \Phi_{\mathrm{af}}$ defines a linear functional $\mathcal{A} \to \mathbb{R}$, hence a hyperplane H_a and a reflection $r_a : \mathcal{A} \to \mathcal{A}$. The *positive* affine roots are those $a \in \Phi_{\mathrm{af}}$ such that $a(\mathfrak{a}) \subseteq \mathbb{R}_{\geqslant 0}$, we denote this set by Φ_{af}^+.

We now choose a special vertex $\mathfrak{x} \in \mathcal{A}$ adjacent to $\mathfrak{a}$. This allows us to identify $\mathcal{A}$ with the vector space $V := X_*(T)_{\Gamma_0} \otimes \mathbb{R}$ by sending $\mathfrak{x}$ to the origin; here, Γ_0 is the absolute Galois group of $\breve{F}$. Each root $a \in \Phi_{\mathrm{af}}$ can then be written as $a = (\alpha, k)$ for a *classical root* $\alpha \in V^*$ and an integer $k \in \mathbb{Z}$ such that $a(v) = \alpha(v) + k$ for all $v \in V$. The reflection $r_a : V \to V$ sends $v \in V$ to $v - a(v)\alpha^\vee = s_\alpha(v) - k\alpha^\vee \in V$.

Of course, following Bruhat–Tits, we would like to associate the *affine root subgroup*

$$U_a = U_\alpha(\varepsilon^k \mathcal{O}_{\breve{F}})$$

with the affine root $a = (\alpha, k)$. Here, $U_\alpha : \breve{F} \to G(\breve{F})$ denotes the root subgroup associated with $\alpha \in \Phi$.

The group $N_G(T)(\breve{F})$ acts on $\mathcal{A}$ by affine–linear maps, with the normal subgroup $T_0 := T(\breve{F}) \cap I \lhd N_G(T)(\breve{F})$ being the kernel of this action. We denote the quotient as $\widetilde{W} = N_G(T)(\breve{F})/T_0$, called the extended affine Weyl group. Similarly, the group $W := N_G(T)(\breve{F})/T(\breve{F})$ acts on V by linear maps, this is the finite Weyl group. Each element in $x \in \widetilde{W}$ can be written as a composition $x = w\varepsilon^\mu$, where $\mu \in X_*(T)_{\Gamma_0}$ and ε^μ denotes the map $v \mapsto v + \mu$ for $v \in V = X_*(T)_{\Gamma_0} \otimes \mathbb{R}$. In this sense, the affine reflection $r_{(\alpha,k)}(v) = s_\alpha(v) - k\alpha^\vee$ coincides with the element $s_\alpha \varepsilon^{k\alpha^\vee} \in \widetilde{W}$.

In the aforementioned articles, we always choose as a dominant chamber in V the chamber which *contains* the alcove $\mathfrak{a}$ under this identification. Let us call this convention the *same–directional* convention (since the dominant chamber and the base alcove both point in the same direction, as viewed from the origin). In the literature, one may just as frequently see the *opposite–directional* convention, where the *anti–dominant* chamber contains the base alcove. Regarding the aforementioned articles, we would like to suggest that the reader follows one of the following advices:

(1) Treat the articles as written. If $\mu \in X_*(T)$ denotes a cocharacter, seen as a function $\mu : \breve{F}^\times \to T(\breve{F})$, then we can consider the torus element $\mu(\varepsilon) \in T(\breve{F})$. It satisfies

$$\mu(\varepsilon) U_\alpha(\varepsilon^k \mathcal{O}_{\breve{F}}) \mu(\varepsilon)^{-1} = U_\alpha(\varepsilon^{k+\langle \mu, \alpha \rangle} \mathcal{O}_{\breve{F}}).$$

Thus, the element in the extended affine Weyl group $\widetilde{W} = N_G(T)(\breve{F})/T_0$ corresponding to $\mu(\varepsilon) \in N_G(T)(\breve{F})$ will be denoted $\varepsilon^{-\mu} \in \widetilde{W}$. In particular, this means our Kottwitz and Newton point maps (e.g. sending $\varepsilon^\mu \in \widetilde{W}$ to $(\mu^{\mathrm{dom}}, \mu \bmod \mathbb{Z}\Phi^\vee)$) are the $(-w_0)$-twisted version of what one might expect. These somewhat confusing circumstances are the reason to write this note.

(2) The reader may associate the root hyperplane $H_{(\alpha,k)} = \{v \in V \mid \alpha(v) + k = 0\} \subseteq V$ with the root subgroup $U_{-\alpha}(\varepsilon^k \mathcal{O}_{\breve{F}})$, essentially negating the convention of Bruhat–Tits (without seriously harming their foundational work). In the article of semi–infinite orbits, one would have to replace the root subgroups U_β by $U_{-\beta}$ everywhere, and the unipotent radical $U(\breve{F})$ by its opposite ${}^{w_0}U(\breve{F})$. With this change, it becomes true that $\varepsilon^\mu \in \widetilde{W}$ is represented by $\mu(\varepsilon) \in T(\breve{F})$.

(3) The reader may take any of the aforementioned articles and mentally perform following changes throughout:

- In the introduction of such an article, instead of adopting the same–directional convention, one chooses the opposite–directional convention.
- Whenever round parenthesis are used to specify affine roots, such as $a = (\alpha, k)$, one thinks of the first coordinate as negated, and would read this example like $a = (-\alpha, k)$.
- Whenever the map $\mathrm{cl} : \Phi_{\mathrm{af}} \to \Phi$ is used in a formula, assigning each affine root to its classical part, one thinks of the input (or equivalently output) of this map as inverted.

Under these changes, a formula such as $\mathrm{cl}(\alpha, k) = \alpha$ would become $\mathrm{cl}(-\alpha, k) = -\alpha$. The formula describing the action of the extended affine Weyl group on Φ_{af}^{+} becomes

$$w\varepsilon^{\mu}(-\alpha, k) = (-w\alpha, k - \langle \mu, \alpha \rangle),$$

which can be written more briefly as

$$w\varepsilon^{\mu}(\alpha, k) = (w\alpha, k + \langle \mu, \alpha \rangle).$$

Hence, $\mu(\varepsilon) \in T(\breve{F})$ represents $\varepsilon^{\mu} \in \widetilde{W}$ after these changes.

The statements of theorems and lemmas need to be adapted for these changes, though only a very small number of results is affected.